Strategy Wars

Mathematics Club, $C\Phi$

September 14th 2023

1 A primer on Dice Questions

a) What is the probability when two dice A and B are rolled, die A will show a number strictly greater than the die B?

b) Three dice are rolled together. What is the probability of getting a sum greater than or equal to 6?

c) There is a higher probability of getting 3 consecutive sixes in 3 throws than getting sum of numbers equal to 10 in 3 throws. Is this statement true or false?

d) I throw a die and keep rethrowing it until I get an odd number. What is the probability that I make 3 or more consecutive throws?

2 A chance at redemption

2c plays a game of dice. The game goes as follows: 2c rolls the die once. His score is what he rolls.

Now, the Grand Master offers him a chance: if 2c is not happy with his roll, he can roll the die again. His old score will be ignored and his new score will be what he rolls. This re-rolling is not compulsory.

What does 2c do?

The *optimal* strategy in this case is as follows: 2c rolls the die once. If he rolls a 1, 2, or a 3, he's not happy with it and re-rolls, while he's perfectly happy with a 4, 5, or a 6 and does not re-roll.

a) What is the expected score if the optimal strategy is used?

b) Now, the grand master says he can re-roll one more time (there are three rolls in total, including the first one). What is the optimal strategy, i.e. when should you re-roll?

c) For the above question, what is the expected value?

d) Now, we roll for a total of n times. We see that as n gets larger and larger, you can keep re-rolling until you get a 6 (and your expected score keeps approaching 6). What is the minimum value of n, for which your strategy is to "re-roll until you get a 6"?

3 Stones Game

There is a single pile of n stones. There are two players, Krishna and Skandan. In each turn, they can remove $1, 2$ or 3 coins. Krishna goes first. The player who removes the last coin wins.

A sample game with $n = 7$: Krishna removes 3 coins, now there are 4 coins remaining. Skandan removes 2 coins, now there are 2 coins removing. Krishna removes 2 coins and wins the game.

It turns out that there exists an optimal strategy, and if both players play optimally, the winner depends on the value of n only. Find the winner for n from 1 to 10. Write "K" if Krishna wins and write "S" if Skandan wins.

Hint: Start from $n = 1$ and see what happens.

4 Stones Game: 2

There is a single pile of n stones. There are two players, Krishna and Skandan. In each turn, they can remove $1, 3, 4$, or 5 coins. Krishna goes first. The player who removes the last coin wins.

A sample game with $n = 14$: Krishna removes 3 coins, now there are 11 coins remaining. Skandan removes 5 coins, now there are 6 coins removing. Krishna removes 1 coin, now there are 5 coins removing. Skandan removes all 5 coins and wins the game!

It turns out that there exists an optimal strategy, and if both players play optimally, the winner depends on the value of n only. Find the winner for n from 11 to 20. Write "K" if Krishna wins and write "S" if Skandan wins.

Hint: Start from $n = 1$ and see what happens.

5 Nim

We're given n piles of stones. Karthikeya and Atreya play the game, with Karthikeya going first. In each turn, they can remove how many ever stones they want from one particular pile. Atreya and Karthikeya can remove coins from the same pile if they want to. The configuration $(10, 5, 3)$ means there are 3 piles: one with 10 stones, one with 5 stones, and one with 3 stones.

For the given configurations, who wins? The number of stones in each pile is represented with brackets. If Karthikeya wins, write "K" and if Atreya wins, write "A".

a) (10)

b) $(1, 1)$

c) $(10, 1)$

d) $(10, 10)$

e) $(10, 15)$

f) $(10, 5, 6)$

g) $(10, 5, 6, 9)$

6 The Dacoit-Merchant Problem

There are n *dacoits* and 1 *merchant* in Dholakpur. *Chota Bheem* has placed an enchantment such that if a Dacoit robs a *merchant* he will become a Merchant himself, however if two or more *dacoits* rob a *merchant*, the *dacoit* who will first rob the *merchant* will become a *merchant* (nothing happens to the other dacoits.). *Dacoits* would love to become a *merchant* but do not want to get robbed after they become a merchant. All *merchants* of Dholakpur are intelligent. Assume that the *dacoits* are equally skilled in robbing people. Will the initial merchant be robbed or not? Comment for general n.

7 Bumble

Consider the number line of integers from 0 to n. Some points on the number line have coins on them. Gorlaid and Lex play a game, in which Gorlaid goes first. In each turn, exactly one coin can be moved to any point to its left on the number line. Note that one coin has to be moved to the left, i.e. you cannot skip a turn. Once all the coins are at $x = 0$, the game ends and the player who moved the last coin wins. Assuming they both play optimally, who wins for the below configurations?

a) There is a single coin at $x = 1$

b) There are two coins, one at $x = 5$ and the other at $x = 8$.

c) There are 6 coins, at $x = 1, 2, 5, 7, 10, 11$.

8 Game of Queens

A queen in a chessboard can move horizontally, vertically, or diagonally. Let's change it up a bit. We have a chessboard, and the Queen can go any number of squares to the left, down, or to the bottom-left diagonally only. Both players sit on the same side of the chessboard. In our chessboard the tiles are labeled as an ordered pair (a, b), starting from $(0, 0)$. Sreejaa and 2C take turns in playing the game, with Sreejaa going first. The person who moves the queen first to the point $(0, 0)$ wins.

It turns out that the winner depends only on the starting position. For instance, if the Queen was at (m, m) or $(0, m)$ or $(m, 0)$ initially, Sreejaa will always win because she can move it to $(0, 0)$ in one step.

For what all starting positions (a, b), $a \leq 5, b \leq 5$ does Sreejaa win?

9 Simple Dice Game

Suppose we play a game with a die where we roll and sum our rolls. We can stop any time and take the sum as our score, but if we roll a face we've rolled before then we lose everything (i.e. our score becomes 0) and stop the game. What strategy will maximize our expected score?

10 More Dice Games

Consider a game on creating a two-digit number in which the player rolls the die once and decides which of the two digits they want that roll to represent, i.e. "ones" or "tens" digit. Then, the player rolls a second time and this determines the other digit. For instance, the player might roll a 5, and decide this should be the "tens" digit, and then roll a 6, so their resulting number is 56.

What strategy should be used to create the largest number on average, i.e. what strategy should be used to get the greatest expected value of a number?

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1

a) If B is 1, A can be any number from 2 till 6. If B is 2, A can be any number from 3 till 6. Keep going on, until we say that B is 6, there's no valid A. Summing up, we get the required probability to be $\frac{15}{36}$.

b) Required probability is 1 - probability of getting a sum of 3, 4, 5 $= 1 - \frac{10}{216} = \frac{206}{216}$.

c) The statement is false. The former probability is $\frac{1}{216}$, while the latter is a bit more (you can calculate it as an exercise).

d) For the condition to be true, my first throw and my second throw both need to be an even number. The probability that this happens is $\frac{1}{4}$.

2

a) The expected score is $\frac{1}{6}(4+5+6) + \frac{3}{6}(3.5) = 4.25$.

b) In two rolls, your expected score is 4.25. So in the first roll, if you get a number > 4.25, stick with it. Else, roll again. Now if you did reroll, in this second throw, if you get a number > 3.5, stick with it, otherwise re-roll.

c) The expected value is $\frac{1}{6}(5+6) + \frac{4}{6}(4.25) = 4.666\ldots$.

d) By the same token, $n = 4$ gives us an expected value of ≈ 4.94, while for $n = 5$ we get a nice expected value of $n = 5.12$. So, $n = 5$ is the answer!

3

The strategy goes like this: if the number is a multiple of 4, if Krishna removes i stones, Skandan removes $4 - i$ stones, so eventually Skandan will remove the last stone. If the number is not a multiple of 4, Krishna can remove enough stones to make the new number divisible by 4, so Krishna wins.

So, the general solution is that for n is divisible by 4, Skandan wins, else Krishna wins.

4

The strategy is similar to the third question. Let us define *good* states as states which the first player wins if he starts from it and *bad* states as states which the first player loses if he starts from it. We observe that if the first player can make the second player reach a *bad* state, he wins and therefore he is in a *good* state. We also observe that every state is either *good* or *bad*. Now, let's try to identify these states. Clearly 1, 3, 4 and 5 are *good* states and 2 is a *bad* state. Now, we can recursively label the other states. n is a good state if one of $n-1$, $n-3$, $n-4$ and $n-5$ is a bad state.

2, 8, 10, 16, 18 etc are bad states. So if a state is of the form $8k + 2$ or $8k$, it is a bad state and other states are good states.

Observe that a pattern is formed and this pattern repeats every 8 states.

5

Check out this link!

Based on this, the solutions are as follows:

a) Karthikeya

b) Atreya

c) Karthikeya

d) Atreya

e) Karthikeya

f) Karthikeya

g) Atreya

6

If there is 1 dacoit only, he will rob the merchant because he himself will become a merchant and put peace. The merchant will be robbed.

If there are 2 dacoits, both of them knows that if one robs the merchant, the other dacoit will rob him. So, the merchant will not be robbed.

Going by the same pattern, if there are even number of dacoits the merchant will survive, else he will be robbed.

7

If you notice carefully, the game is the same as the nim game shown above! Once you observe this, our job becomes peaceful. Based on this, the solutions are just to take the bitwise xor of the given values as follows:

a) Gorlaid

b) Gorlaid

c) Lex

8

This is an amazing problem in game theory, you can read about it here: Wythoff's Game

Based on this, the positions $(1, 2), (2, 1), (3, 5), (5, 3)$ will be won by 2C, any other position within the range will be won by Sreejaa.

9

We want to roll until the expected sum after rolling is less than our current sum. Let C be our current sum and S be the set of faces that have been rolled already. Then we should stop if

$$\frac{|S|}{6} \cdot 0 + \sum_{i \notin S} \frac{1}{6}(C+i) < C.$$

Using the fact that $\sum_{i \in S} i = C$, this inequality simplifies to

$$C(|S|+1) > 21.$$

After our first roll, $|S| = 1$ and $C < 6$ so $C(|S|+1) \leq 12$. Hence we should roll again. After our second roll, $|S| = 2$, so we should stop if $C > 7$. After our third roll, $|S| = 3$, so we should stop if $C > \frac{21}{4}$, that is, if $C \geq 6$. However, if we have made it to our third roll, C must be at least 6, and so we should stop at this point.

Thus: Roll twice. If the second roll is not the same as the first, and the sum is less than 7, roll again and stop; otherwise, stop.

10

A strategy in this game is merely a rule for deciding whether the first roll should be the "tens" digit or the "ones" digit. If the first roll is a 6, then it must go in the "tens" digit, and if it's a 1, then it must go in the "ones" digit. This leaves us with what to do with 2,3,4 and 5. If the first roll is b, then using it as the "ones" digit results in an expected number of $\frac{7}{2} \cdot 10 + b$. Using it as the "tens" digit results in an expected number of $10b + \frac{7}{2}$. So, when is $10b + \frac{7}{2} > \frac{7}{2} \cdot 10 + b$? When $b \geq 4$. Thus, if the first roll is 4, 5 or 6, the player should use it for the "tens" digit. With this strategy, the expected value of the number is

$$\frac{1}{6}(63.5 + 53.5 + 43.5 + 38 + 37 + 36) = 45.25.$$