Problem Set 1

Mathematics Club, IITM

3rd November 2022

1 Problems from the Presentation

1. For the configuration of Bishops in a chessboard, we found that the maximum number of Bishops $n \times n$ chessboard was $2n - 2$. In how many ways can you do this?
2. For the configuration of Knights in a chessboard, how many knights can you place in a $n \times n$ chessboard and in how many ways?
3. Prove that $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$
4. Solving for Lucas Numbers: Take $F_0 = 2, F_1 = 1, F_k = F_{k-1} + F_{k-2}, k \geq 2$

   (Brownie Points for finding out its relationship with Fibonacci Numbers)
5. Prove Strong induction using regular induction. While using strong induction, is one base case always enough, or do you need more?

A sidenote: We obtained the chessboard problems from Challenging Mathematical Problems with Elementary Solutions

2 More Problems

1. Differential Equations using Matrices: Just like the Fibonacci Sequence, it's possible to solve differential equations using matrices. This arises from an interesting definition: For a matrix A, define

$$e^A = I + A\frac{1}{1!} + A^2\frac{1}{2!} + A^3\frac{1}{3!} + \dots$$

   Here, we can write $A = PDP^{-1}$, for a diagonalizable matrix A.

   Using this, solve the differential equation for simple harmonic oscillator:

$$\frac{d^2x(t)}{dx^2} + w^2x(t) = 0$$

We'll sign off with a hint about the integration bee: Feynman Integration!

Problem Set 2

Mathematics Club, IITM

27th November 2022

1 Problems

1.1 A General Problem

Five-digit numbers are formed using digits 1, 4, 5, 6 and 9 without repetition. What is the sum of all such possible numbers?

1.2 Co-prime integers

Define the function $\phi(n)$ to be the number of integers co-prime to and lesser than n.

- What is $\phi(p)$, where p is prime?
- What is $\phi(z)$, such that $z = p^k$ for some prime p?
- Come up with the general formula for $\phi(n)$ for arbitrary n. Is this even possible?

1.3 L_p metrics

Background: Consider 2-D Euclidean space, with the distance between two points to be defined as $D((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Now, a circle is the set of all points equidistant from a given point (called the *centre*). The value of π is, of course, the ratio of the circumference to the diameter, which happens to be 3.14.

The distance between two points in L^p space is defined to be:

$$\|x\|_p = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{\frac{1}{p}}$$

Indeed, our familiar Euclidean space is just the case $p = 2$.

If $p = 1$,

$$D((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Using our new definition of the distance, **what figure would be the set of all points equidistant from a given point? What would be the value of π_p (i.e. the ratio of circumference to the diameter for the new figure) in this case?**

It is certainly possible to come up (numerically) with a value of the circumference to the diameter, for every value of π_p for every p in the L^p metric. **What interesting observations do we get in that case?**

Wikipedia Article

1.4 Similarity between addition and multiplication

Consider the set of Real numbers, $\mathbb{R}$ under addition. We can observe a few nice properties:

1. If you add two real numbers, you end up with a real number
2. $a + (b + c) = (a + b) + c$
3. There is a real number 0, such that $a + 0 = 0 \forall a$
4. There is an additive inverse for every element, i.e. for every a, there exists some b such that $a + b = 0$. b is, of course, the negative of a.

This is the notion of a mathematical group. A mathematical group is a set which satisfies four conditions, under some operation:

1. Closure
2. Associativity
3. Existence of identity
4. Existence of inverse

Prove that $\mathbb{R}_+$ is a group under multiplication.

Further, notice a few interesting properties:

1. $f(x) = e^x$ maps $\mathbb{R}$ to $\mathbb{R}_+$
2. $\mathbb{R}$ is a group under addition
3. $\mathbb{R}_+$ is a group under multiplication
4. $e^{x+y} = e^x \times e^y$

Imagine you have the set $\mathbb{R}$, with addition as the operation. If you apply the function $f(x) = e^x$ on the group, we end up with R_+, and addition *converts to multiplication*.

This is the idea of a Group Isomorphism.

A group isomorphism is a function f from a group G with an operation + to a group H with an operation $\times$, such that $f(u + v) = f(u) \times f(v)$

Prove that the group $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}_+$ under multiplication.

2 A Footnote

We'll sign off with a hint about the integration bee: Learn about the Gaussian Integral!