DC Session 1
Counting Principles and Distributions

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Outline

We'll learn all these!

Bijection

- Counting Principles
- Distributions

Problem Statement

In an election, candidate A has p and B has q votes. WLOG $p > q$. What is the probability that A stays ahead of B throughout the counting process?

Example for $p = 3, q = 2$:

10

A A A B B

A A B A B

A A B B A

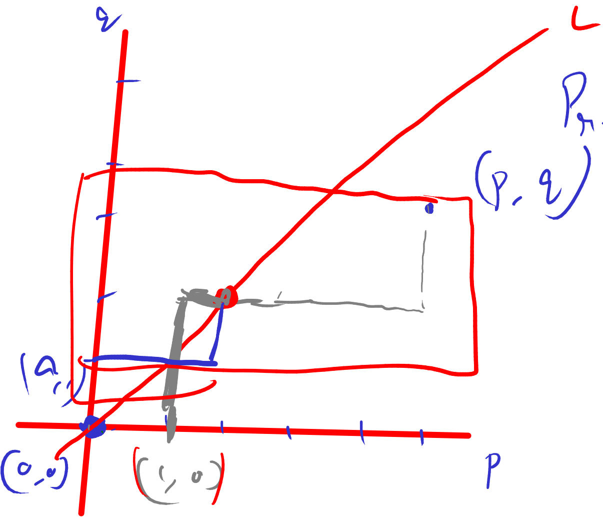
A B A B A

$\frac{2}{10}$

$^5C_2$

Check here:

$x = y$



Assume first step

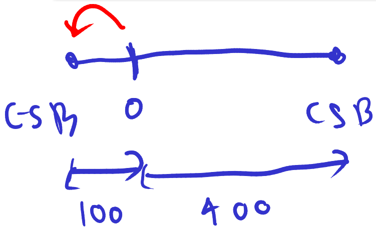
$$Pr = \frac{{}^{p+q-1}C_p}{{}^{p+q-1}C_q} = \frac{q}{p}$$

$$Ans = \left(\frac{p}{q+p}\right)\left[1-\frac{q}{p}\right] = \frac{p-q}{p+q}$$

Intro to Random Walks (more in sesh 3)

Random Walk

Drunk Haricharan is equally likely to move front or back, moving 1m each step. The road from ESB to CSB is a straight road of length 500m. Once he reaches ESB or CSB, he passes out. If he starts 100m from ESB, what is the probability that he passes out in ESB? α



$$E[X_i] = \sum_x x\,p(x)$$

$X_i \to \begin{cases} +1 \to \text{front} \\ -1 \to \text{Back} \end{cases}$ ith step

$$E[X_i] = 0$$

$$E\Big[\underbrace{\sum_{i=1}^{\infty} X_i}_{\text{displacement}}\Big] = 0 = -\alpha\, 100 + (1-\alpha)\, 400$$

$$\alpha = \frac{4}{5}$$

Probability Distributions

- Binomial Distribution
- Normal Distribution
- Poisson Distribution

Assuming knowledge of random variables, continuous and discrete random variables, expectation, mean, variance, probability distribution function.

r.v = X

$$E[X] = \boxed{\sum_{x \in X} x\, p(x)}$$

$$E[g(X)] = \sum_{x \in X} g(x)\, p(x) \to \int_{x=-\infty}^{\infty} g(x)\, p(x)\, dx$$

$$\mathrm{Var}(x) = E[x^2] - (E[x])^2 = E[(x - E[x])^2]$$

Exponential Distribution

Let X be a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x^{-2} & \text{for } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}(X) = (2\log 2)$ and $\mathrm{Var}(X) = (2 - (2\log 2)^2.)$

$$\int_1^2 \underbrace{x}_{g(x)} \underbrace{\left(2x^{-2}\right)}_{p(x)} dx = \boxed{2\ln 2}$$

$$E[x^2] = \int_1^2 x^2 \, 2x^{-2} = \boxed{2}$$

$$\boxed{\mathrm{Var}(x) = 2 - (2\ln 2)^2}$$

Binomial $B(p, N)$: $P(r) = \binom{N}{r} p^r (1-p)^{N-r}, \boxed{r \in 0, 1, 2, \ldots, N}$

Probability of r victories when you toss a $\underbrace{\text{biased coin}}_{p}$ N times

$$\sum_{r=0}^{N} p(r) = (p + 1 - p)^N = 1^N = \boxed{1}$$

$$\boxed{R = X_1 + X_2 + \cdots X_N}$$

$X_i \to$ $1 \to$ Head, $0 \to$ Tail

in ith toss

Linearity of Expectation, Mean, and Variance

$E[x+y] = E[x] + E[y]$

$E[\alpha x] = \alpha E[x]$

$$E[\sum_i c_i X_i] = \sum_i c_i E[X_i]$$

$X_i \to$ behav

$E[x_i] = p$

Mean:

$E[R] = \sum_{r=0}^{N} r \binom{N}{r} p^r (1-p)^{N-r} = \sum_{i=0}^{N} E[x_i] = \boxed{Np}$

Variance:

$E[x_i^2] = p$

$Var(x) = \sum_i E[x_i^2] - (E[x_i])^2 = N[p - p^2] = Np(1-p)$

Normal Distribution

$e^{-\frac{x^2}{2}}$

Normal Distribution is defined:

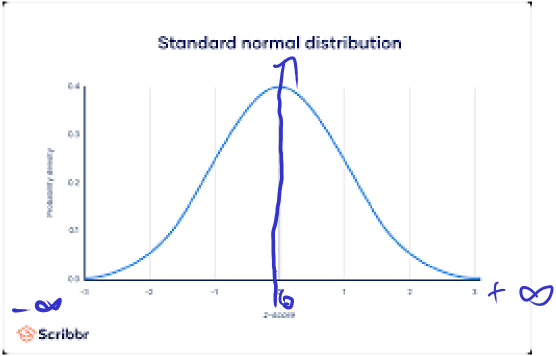
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in (-\infty, \infty)$$

- Symmetric about $x = \mu$
- μ is the Mean (average)
- σ^2 is the Variance

$\mu = 0, \sigma = 1$ *is called the standard normal distribution.*

Plotting the standard normal distribution

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



Why is the normal distribution useful?

Enter Central Limit Theorem (read up), which says something roughly like if you have enough variables, most things in the world tend to behave like the Normal distribution. This is very interesting!

Many things, say the errors in some measurement, are usually Gaussian.

Poisson Distribution

$\lambda \to$ parameter

$$P(n) = e^{-\lambda}\frac{\lambda^n}{n!}, n \in N$$

Compute the mean, variance, $\sum_{n=0}^{\infty} P(n)$.

More Fun Problems

Linearity of Expectation

- You have n foxes and m hounds running from a tunnel. What is the expected number of foxes that follow hounds? (Click for Intuition)
- You have m single men and w single women seated around a round table. Ancient traditions state that a man to the left of a woman might are a marriageable couple. What is the expected number of marriageable couples?
- An n-term sequence in which each term is either 0 or 1 is called a binary sequence of length n. Let an be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let bn be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. The value of $\frac{b_{2023}}{a_{2022}}$ is: